

# Preservation of physical properties of stochastic Maxwell equations with additive noise via stochastic multi-symplectic methods



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## ARTICLE INFO

### Article history:

Received 13 December 2014

Received in revised form 3 September 2015

Accepted 24 November 2015

Available online 2 December 2015

### Keywords:

Stochastic Maxwell equations

Stochastic Hamiltonian partial differential equations

Dissipative property of averaged energy

Conservation law of averaged divergence

Stochastic multi-symplectic method

## ABSTRACT

Stochastic Maxwell equations with additive noise are a system of stochastic Hamiltonian partial differential equations intrinsically, possessing the stochastic multi-symplectic conservation law. It is shown that the averaged energy increases linearly with respect to the evolution of time and the flow of stochastic Maxwell equations with additive noise preserves the divergence in the sense of expectation. Moreover, we propose three novel stochastic multi-symplectic methods to discretize stochastic Maxwell equations in order to investigate the preservation of these properties numerically. We make theoretical discussions and comparisons on all of the three methods to observe that all of them preserve the corresponding discrete version of the averaged divergence. Meanwhile, we obtain the corresponding dissipative property of the discrete averaged energy satisfied by each method. Especially, the evolution rates of the averaged energies for all of the three methods are derived which are in accordance with the continuous case. Numerical experiments are performed to verify our theoretical results.

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## 1. Introduction

In modeling of physical phenomena, stochastic differential equations are required to quantify the effects of randomness on the mathematical model. Taking the context of electromagnetism as an example, to model precise microscopic origins of randomness (the thermal motion of electrically charged microparticles), [12] established the theory of fluctuations of an electromagnetic field, which at the level of macroscopic view was via introducing fluctuation sources to obtain stochastic Maxwell equations. Based on this model, [14] proposed a method based on Wiener chaos expansion to determine the near field thermal radiation, and [10] described the fluctuation of the electromagnetic field using spectral representation. Without modeling the precise origins of randomness, rather assume that they lead to small stochastic variations of the coefficients of the equations, [7] studied the propagation of ultra-short solitons in a cubic nonlinear media, which is modeled by nonlinear Maxwell equations with stochastic variations of media; and assume that the externally imposed source is a random field, which is expressed by a Q-Wiener process, [4,9,11] dealt with the mathematical analysis of stochastic problems arising in the

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<sup>1</sup> Authors are supported by the NNSFC (Nos. 91130003, 11021101 and 11290142). The third author is also supported by the NNSFC (Nos. 11301001 and 20135QRL030ZD).

theory of electromagnetic in complex media, including well posedness, controllability and homogenization. The stochastic model considered in this paper is based on [11, Chapter 12] for the isotropic homogeneous medium with an external source.

Recently, the stochastic multi-symplectic structure for three dimensional (3-D) stochastic Maxwell equations with additive noise was proposed in [2], based on the stochastic version of variational principle, which means that stochastic Maxwell equations are a system of stochastic Hamiltonian partial differential equations (PDEs). It has been widely recognized that the structure-preserving numerical methods have the remarkable superiority to conventional numerical methods when applied to Hamiltonian ODEs and PDEs, such as long-term behavior, structure-preserving, physical properties-preserving (energy, divergence, charge) etc.; see [6] and the references therein. Efforts have been devoted to the stochastic case. For example, authors in [5] established the theory for the stochastic multi-symplectic conservation law for the stochastic Hamiltonian PDEs and investigated a stochastic multi-symplectic method for stochastic nonlinear Schrödinger equation. Also a stochastic multi-symplectic wavelet collocation method was proposed in [3] to approximate stochastic Maxwell equations with a class of multiplicative noise, while [2] proposed another stochastic multi-symplectic method based on the stochastic variational principle.

Different from the approach of reference [2], we use a direct way to represent the stochastic Maxwell equations as another system of stochastic Hamiltonian PDEs, which avoids introducing extra variables and leads to cost efficiency. As a result, the stochastic Maxwell equations preserve the stochastic multi-symplectic conservation law almost surely. Meanwhile, we show that the averaged energy increases linearly as the growth of time, with the rate being  $K = 3(\lambda_1^2 + \lambda_2^2)Tr(Q)$ . Here  $\lambda_1$  and  $\lambda_2$  represents the levels of noise, and  $Tr(Q)$  denotes the trace of operator  $Q$ . It means that the growth rate depends on the scale of noises and the trace of covariance operator only. This dissipative property of averaged energy may be due to the existence of external source. For the divergence, it is proved that the flow of stochastic Maxwell equations preserves the divergence in the sense of expectation. It means that electric flux and magnetic flux are preserved in Gaussian random fields in the statistical sense. In this paper, we propose three numerical methods to discretize stochastic Maxwell equations with additive noise in order to investigate the preservation of these physical properties numerically. Method-I is based on the application of implicit midpoint method in both temporal and spatial directions to the equivalent stochastic Hamiltonian PDEs, while Method-II being a three-layer method is constructed by central difference in both temporal and spatial directions, which exhibits the grid staggering property of electromagnetism. We utilize central difference in spatial direction and implicit midpoint method in temporal direction to obtain Method-III. We demonstrate that all of the three numerical methods preserve the corresponding discrete versions of multi-symplectic conservation law. Another aim of this paper is to investigate the numerical preservation of some important physical quantities including energy and divergence by numerical methods. For the energy, we obtain the corresponding dissipative property of the discrete averaged energy satisfied by each method. Furthermore, utilizing the adaptedness of solutions to stochastic Maxwell equations and properties of Wiener process, we estimate the dissipative rates with respect to time for three methods in our consideration, and we show that the discrete averaged energies evolve at most linearly with respect to time under certain assumptions. As for divergence, we show that all of the three methods preserve the discrete conservation law of averaged divergence well, as shown theoretically in Theorems 3.4, 3.8 and 3.12. Finally, numerical experiments are performed to validate the theoretical results.

The outline of this paper is as follows. In section 2 we present some preliminaries about stochastic Maxwell equations, including theorems about the evolution of energy and divergence, and the intrinsic stochastic multi-symplectic structure. Section 3 is devoted to the comparison and analysis of three stochastic multi-symplectic numerical methods in the aspect of averaged energy and averaged divergence. Numerical experiments for stochastic Maxwell equations with additive noise are performed in section 4 to verify our theoretical results. Finally, concluding remarks are given in Section 5.

In the sequels, we let  $\mathbf{e} = (1, 1, 1)^T$  and denote by  $\langle \cdot, \cdot \rangle_{L^2}$  the  $L^2(\Theta)$  inner product, by  $\langle \cdot, \cdot \rangle$  the Euclidean inner product, by  $|\cdot|$  the Euclidean norm, and by  $\mathcal{E}$  the expectation.

## 2. Stochastic Maxwell equations with additive noise

It is of interest to study phenomena where the densities of the electric and magnetic currents are assumed to be stochastic. These can be modeled by the following 3-D stochastic Maxwell equations with additive noise

$$\begin{cases} \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \lambda_1 \mathbf{e} \dot{\chi} & \text{in } (0, T) \times \Theta, \\ \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} + \lambda_2 \mathbf{e} \dot{\chi} & \text{in } (0, T) \times \Theta, \end{cases} \quad (2.1)$$

with initial conditions

$$\begin{aligned} \mathbf{E}(0, x, y, z) &= (E_{10}, E_{20}, E_{30}) \text{ in } \Theta, \\ \mathbf{H}(0, x, y, z) &= (H_{10}, H_{20}, H_{30}) \text{ in } \Theta, \end{aligned} \quad (2.2)$$

and perfectly electric conducting (PEC) boundary conditions

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } (0, T] \times \partial\Theta, \quad (2.3)$$

where  $T > 0$ ,  $\Theta$  is a bounded and simply connected domain in  $\mathbb{R}^3$  with smooth boundary  $\partial\Theta$ ,  $\mathbf{n}$  represents the unit outward normal of  $\partial\Theta$ , and  $\lambda_1, \lambda_2$  are real numbers representing the scales of noise. It is convenient at this point to give a precise mathematical definition of  $\dot{\chi}$ .

Hereafter, let  $W$  be a Q-Wiener process defined on a given probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , with values in the Hilbert space  $L^2(\Theta)$ , which is a space of square integrable real-valued functions. Let  $\{e_m\}_{m \in \mathbb{N}}$  be an orthonormal basis of  $L^2(\Theta)$  consisting of eigenvectors of a symmetric, nonnegative and finite trace operator  $Q$ , i.e.,  $Tr(Q) < \infty$  and  $Qe_m = \eta_m e_m$ . Then there exists a sequence of independent real-valued Brownian motions  $\{\beta_m\}_{m \in \mathbb{N}}$  such that

$$W(t, x, y, z, \omega) = \sum_{m=1}^{\infty} \sqrt{\eta_m} e_m(x, y, z) \beta_m(t, \omega), \quad t \geq 0, (x, y, z) \in \Theta, \omega \in \Omega. \tag{2.4}$$

And formally set  $\dot{\chi} = \frac{dW}{dt}$ .

**Remark 1.** The expansion formula (2.4) of Q-Wiener process  $W(t, x, y, z, \omega)$  is based on the orthonormal basis of  $L^2(\Theta)$ , which separates the variables  $(x, y, z)$  and  $(t, \omega)$  apart. We note that the well known Wiener chaos expansion (WCE) separates the variable  $\omega$  apart from other temporal or spacial variables, and if we apply WCE to the sequence of Brownian motions  $\{\beta_m(t, \omega)\}_{m \in \mathbb{N}}$ , we may also use WCE method to approximate the original equations (2.1); see [1] for more details about Wiener chaos expansion.

We refer interested readers to [9] for the well-posedness of problem (2.1). The authors present some results on stochastic integrodifferential equations in Hilbert spaces, motivated from and applied to problems arising from the mathematical modeling of electromagnetics fields in complex random media. They examine the mild, strong and classical well-posedness for Cauchy problem of the integrodifferential equation which describes Maxwell equations complemented with the general linear constitutive relations describing such media, with either additive or multiplicative noise.

2.1. Dissipative property of averaged energy

In this subsection, we consider the property of averaged energy for system (2.1). The following theorem shows that the averaged energy evolves linearly with respect to time  $t$  and with a growth rate  $K = 3(\lambda_1^2 + \lambda_2^2)Tr(Q)$ , here  $Tr(Q)$  denotes the trace of operate  $Q$ , i.e.,  $Tr(Q) = \sum_{m \in \mathbb{N}} \langle Qe_m, e_m \rangle_{L^2} = \sum_{m \in \mathbb{N}} \eta_m$ .

**Theorem 2.1.** Let  $\mathbf{E}$  and  $\mathbf{H}$  be the solutions of the equations (2.1)–(2.3). Then for  $t \in [0, T]$ , there exists a constant  $K = 3(\lambda_1^2 + \lambda_2^2)Tr(Q)$  such that the averaged energy satisfies the following dissipative property

$$\mathcal{E}(\Phi^{exact}(t)) = \mathcal{E}(\Phi^{exact}(0)) + Kt, \tag{2.5}$$

where  $\Phi^{exact}(t) = \int_{\Theta} (|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2) dx dy dz$ .

**Proof.** We write (2.1) into

$$d\mathbf{E} = \nabla \times \mathbf{H} dt - \lambda_1 \mathbf{e} dW, \\ d\mathbf{H} = -\nabla \times \mathbf{E} dt + \lambda_2 \mathbf{e} dW.$$

Let

$$F_1(\mathbf{E}(t)) = \int_{\Theta} |\mathbf{E}(t)|^2 dx dy dz \quad \text{and} \quad F_2(\mathbf{H}(t)) = \int_{\Theta} |\mathbf{H}(t)|^2 dx dy dz.$$

Since  $F_1$  and  $F_2$  are Fréchet derivable, the derivatives of  $F_1$  along direction  $\varphi$  or  $(\varphi, \psi)$  are as follows,

$$DF_1(\mathbf{E})(\varphi) = 2 \int_{\Theta} \langle \mathbf{E}, \varphi \rangle dx dy dz, \\ D^2 F_1(\mathbf{E})(\varphi, \psi) = 2 \int_{\Theta} \langle \psi, \varphi \rangle dx dy dz. \tag{2.6}$$

Applying the infinite dimensional Itô formula to  $F_1(\mathbf{E})$ , we have

$$\begin{aligned}
 F_1(\mathbf{E}(t)) &= F_1(\mathbf{E}(0)) + \int_0^t DF_1(\mathbf{E}(s))(-\lambda_1 \mathbf{e}dW(s)) \\
 &+ \int_0^t \left\{ DF_1(\mathbf{E}(s))(\nabla \times \mathbf{H}(s)) + \frac{1}{2} \text{Tr} \left[ D^2 F_1(\mathbf{E}(s))(-\lambda_1 \mathbf{e}Q^{\frac{1}{2}})(-\lambda_1 \mathbf{e}Q^{\frac{1}{2}})^* \right] \right\} ds.
 \end{aligned} \tag{2.7}$$

Substitute (2.6) into (2.7) leads to

$$\begin{aligned}
 F_1(\mathbf{E}(t)) &= F_1(\mathbf{E}(0)) + 2 \int_0^t \int_{\Theta} \langle \mathbf{E}(s), -\lambda_1 \mathbf{e}dW(s) \rangle dx dy dz \\
 &+ \int_0^t \int_{\Theta} \left\{ 2 \langle \mathbf{E}(s), \nabla \times \mathbf{H}(s) \rangle + 3\lambda_1^2 \sum_{m \in \mathbb{N}} \eta_m e_m^2(x, y, z) \right\} dx dy dz ds.
 \end{aligned} \tag{2.8}$$

Similarly, we apply Itô formula to function  $F_2(\mathbf{H}(t))$  and obtain

$$\begin{aligned}
 F_2(\mathbf{H}(t)) &= F_2(\mathbf{H}(0)) + 2 \int_0^t \int_{\Theta} \langle \mathbf{H}(s), \lambda_2 \mathbf{e}dW(s) \rangle dx dy dz \\
 &+ \int_0^t \int_{\Theta} \left\{ 2 \langle \mathbf{H}(s), -\nabla \times \mathbf{E}(s) \rangle + 3\lambda_2^2 \sum_{m \in \mathbb{N}} \eta_m e_m^2(x, y, z) \right\} dx dy dz ds.
 \end{aligned} \tag{2.9}$$

Summing (2.8) and (2.9) leads to

$$\begin{aligned}
 F_1(\mathbf{E}(t)) + F_2(\mathbf{H}(t)) &= F_1(\mathbf{E}(0)) + F_2(\mathbf{H}(0)) \\
 &+ 2 \int_0^t \int_{\Theta} \left( \langle \mathbf{H}(s), \lambda_2 \mathbf{e}dW(s) \rangle - \langle \mathbf{E}(s), \lambda_1 \mathbf{e}dW(s) \rangle \right) dx dy dz \\
 &+ 2 \underbrace{\int_0^t \int_{\Theta} \left( \langle \mathbf{H}(s), -\nabla \times \mathbf{E}(s) \rangle + \langle \mathbf{E}(s), \nabla \times \mathbf{H}(s) \rangle \right) dx dy dz ds}_{(a)} \\
 &+ 3(\lambda_1^2 + \lambda_2^2) \int_0^t \int_{\Theta} \sum_{m \in \mathbb{N}} \eta_m e_m^2(x, y, z) dx dy dz ds.
 \end{aligned}$$

Using the Green formula and PEC boundary conditions, we get

$$(a) = -2 \int_0^t \int_{\partial\Theta} (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS ds = 0.$$

Hence, there exists a constant  $K = 3(\lambda_1^2 + \lambda_2^2) \text{Tr}(Q)$ , such that

$$\begin{aligned}
 F_1(\mathbf{E}(t)) + F_2(\mathbf{H}(t)) &= F_1(\mathbf{E}(0)) + F_2(\mathbf{H}(0)) + Kt \\
 &+ 2 \int_0^t \int_{\Theta} \left( \langle \mathbf{H}(s), \lambda_2 \mathbf{e}dW(s) \rangle - \langle \mathbf{E}(s), \lambda_1 \mathbf{e}dW(s) \rangle \right) dx dy dz.
 \end{aligned} \tag{2.10}$$

The assertion follows from applying expectation on equation (2.10).  $\square$

2.2. Conservation law of averaged divergence

As is well known that the electric field and magnetic field are divergence-free if the media is lossless in deterministic case. The following theorem shows that for stochastic Maxwell equations with additive noise (2.1) the electric field and magnetic field are still divergence-free, but in the sense of expectation. In the following, assume that  $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$  and  $(V, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$  are two separable Hilbert spaces, and denote  $HS(U, V)$  the space of all Hilbert–Schmidt operators from  $U$  to  $V$ . The norm is defined by

$$\|Q\|_{HS(U,V)} = \left( \sum_m \|Q f_m\|_{V^2} \right)^{\frac{1}{2}}, \quad \forall Q \in HS(U, V),$$

with  $\{f_m\}_{m \in \mathbb{N}}$  being an orthonormal basis of  $U$ .

**Theorem 2.2.** Assume that  $Q^{\frac{1}{2}} \in HS(L^2(\Theta), H^1(\Theta))$  with  $H^1(\Theta)$  being the Sobolev space. Then system (2.1) preserves the averaged divergence, i.e.,

$$\mathcal{E}(\operatorname{div}(\mathbf{E}(t))) = \mathcal{E}(\operatorname{div}(\mathbf{E}(0))), \quad \mathcal{E}(\operatorname{div}(\mathbf{H}(t))) = \mathcal{E}(\operatorname{div}(\mathbf{H}(0))). \tag{2.11}$$

**Proof.** Let

$$G(\mathbf{E}) = \operatorname{div} \mathbf{E} = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}.$$

Since  $G$  is Fréchet derivable, the derivatives of  $G$  along direction  $\varphi$  or  $(\varphi, \psi)$  are as follows,

$$DG(\mathbf{E})(\varphi) = \operatorname{div} \varphi, \quad D^2G(\mathbf{E})(\varphi, \psi) = 0. \tag{2.12}$$

Applying the infinite dimensional Itô formula to  $G(\mathbf{E})$ , we have

$$G(\mathbf{E}(t)) = G(\mathbf{E}(0)) + \int_0^t DG(\mathbf{E}(s))(-\lambda_1 \mathbf{e} dW) + \int_0^t DG(\mathbf{E}(s))(\nabla \times \mathbf{H}(s)) ds. \tag{2.13}$$

Substituting (2.12) into (2.13) and keeping in mind a fact  $\operatorname{div}(\nabla \times \mathbf{Y}) = 0, \forall \mathbf{Y}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we get

$$G(\mathbf{E}(t)) = G(\mathbf{E}(0)) + \int_0^t DG(\mathbf{E}(s))(-\lambda_1 \mathbf{e} dW). \tag{2.14}$$

The first assertion in (2.11) follows from taking the expectation on both sides of (2.14).

Analogously, we can get the second assertion in (2.11), by applying Itô formula to function  $\operatorname{div} \mathbf{H}$ .  $\square$

2.3. Stochastic multi-symplectic conservation law

In this paper, we use a direct way to rewrite equation (2.1) into the form of stochastic Hamiltonian PDEs. Obviously, the direct approach may avoid introducing extra variables; see [2] for another approach based on the stochastic version of variational principle to rewrite equation (2.1). By denoting  $Z = (H_1, H_2, H_3, E_1, E_2, E_3)^T$ , we have

$$Md_t Z + K_1 Z_x dt + K_2 Z_y dt + K_3 Z_z dt = \nabla_Z S_1(Z) dt + \nabla_Z S_2(Z) \circ dW, \tag{2.15}$$

where  $\circ$  in the second term of the right-hand side of (2.15) denotes Stratonovich sense of integral, and skew-symmetric matrices  $M, K_1, K_2, K_3$  are given by

$$M = \begin{pmatrix} 0 & -I_{3 \times 3} \\ I_{3 \times 3} & 0 \end{pmatrix}, \quad K_p = \begin{pmatrix} \mathcal{D}_p & 0 \\ 0 & \mathcal{D}_p \end{pmatrix}, \quad \forall p = 1, 2, 3, \tag{2.16}$$

with  $I_{3 \times 3}$  being the identity matrix and

$$\mathcal{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{D}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{D}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.17}$$

$$S_1(Z) = 0, \quad S_2(Z) = \lambda_2(H_1 + H_2 + H_3) - \lambda_1(E_1 + E_2 + E_3). \tag{2.18}$$

Similar as the proof of [5, Theorem 2.2], we have the following theorem.

**Theorem 2.3.** The stochastic Hamiltonian PDEs (2.15) possess the stochastic multi-symplectic conservative law locally

$$d_t \omega + \partial_x \kappa_1 dt + \partial_y \kappa_2 dt + \partial_z \kappa_3 dt = 0, \text{ a.s.}$$

i.e.,

$$\begin{aligned} & \int_{y_0}^{y_1} \int_{x_0}^{x_1} \int_{z_0}^{z_1} \omega(t_1, x, y, z) dx dy dz + \int_{y_0}^{y_1} \int_{t_0}^{t_1} \int_{z_0}^{z_1} \kappa_1(t, x_1, y, z) dt dy dz \\ & + \int_{x_0}^{x_1} \int_{t_0}^{t_1} \int_{z_0}^{z_1} \kappa_2(t, x, y_1, z) dt dx dz + \int_{x_0}^{x_1} \int_{t_0}^{t_1} \int_{y_0}^{y_1} \kappa_3(t, x, y, z_1) dt dx dy \\ & = \int_{y_0}^{y_1} \int_{x_0}^{x_1} \int_{z_0}^{z_1} \omega(t_0, x, y, z) dx dy dz + \int_{y_0}^{y_1} \int_{t_0}^{t_1} \int_{z_0}^{z_1} \kappa_1(t, x_0, y, z) dt dy dz \\ & + \int_{x_0}^{x_1} \int_{t_0}^{t_1} \int_{z_0}^{z_1} \kappa_2(t, x, y_0, z) dt dx dz + \int_{x_0}^{x_1} \int_{t_0}^{t_1} \int_{y_0}^{y_1} \kappa_3(t, x, y, z_0) dt dx dy, \end{aligned}$$

where  $\omega(t, x, y, z) = \frac{1}{2} dZ \wedge M dZ$ ,  $\kappa_p(t, x, y, z) = \frac{1}{2} dZ \wedge K_p dZ$  ( $p = 1, 2, 3$ ) are the differential 2-forms associated with the skew-symmetric matrices  $M$  and  $K_p$ , respectively, and  $(t_0, t_1) \times (x_0, x_1) \times (y_0, y_1) \times (z_0, z_1)$  is the local definition domain of  $Z(t, x, y, z)$ .

### 3. Stochastic multi-symplectic methods

In this section we mainly focus on the analysis of three stochastic multi-symplectic methods for the stochastic Maxwell equations (2.1), including the dissipative property of the discrete averaged energy and the conservative property of the discrete averaged divergence.

Let  $\Delta x$ ,  $\Delta y$  and  $\Delta z$  be the mesh sizes along  $x$ ,  $y$  and  $z$  directions, respectively, and  $\Delta t$  be the time step length. The temporal-spatial domain we are interested in the following sections is  $[0, T] \times \Theta := [0, T] \times [x_L, x_R] \times [y_L, y_R] \times [z_L, z_R]$ . It is partitioned by parallel lines, where  $t_n = n\Delta t$  and  $x_i = x_L + i\Delta x$ ,  $y_j = y_L + j\Delta y$ ,  $z_k = z_L + k\Delta z$  for  $n = 0, 1, \dots, N$  and  $i = 0, 1, \dots, I$ ;  $j = 0, 1, \dots, J$ ;  $k = 0, 1, \dots, K$ . The grid point function  $Z_{i,j,k}^n$  is the approximation of  $Z(t, x, y, z)$  at node  $(t_n, x_i, y_j, z_k)$ . The general difference operators are employed by:

$$\begin{aligned} \delta_t Z_{i,j,k}^n &= \frac{Z_{i,j,k}^{n+1} - Z_{i,j,k}^n}{\Delta t}, & \bar{\delta}_t Z_{i,j,k}^n &= \frac{Z_{i,j,k}^{n+1} - Z_{i,j,k}^{n-1}}{2\Delta t}, \\ \delta_x Z_{i,j,k}^n &= \frac{Z_{i+1,j,k}^n - Z_{i,j,k}^n}{\Delta x}, & \bar{\delta}_x Z_{i,j,k}^n &= \frac{Z_{i+1,j,k}^n - Z_{i-1,j,k}^n}{2\Delta x}. \end{aligned} \tag{3.1}$$

The same definitions hold for operators  $\delta_y$ ,  $\bar{\delta}_y$ ,  $\delta_z$ ,  $\bar{\delta}_z$ .

#### 3.1. Method-I

Method-I is derived by applying the implicit midpoint method both in spatial and temporal directions to the equations (2.15), similarly as the approach in [2], but for the different form of stochastic Hamiltonian PDEs for equations (2.1). It is stated as follows

$$\begin{aligned} & M \delta_t Z_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}^n + K_1 \delta_x Z_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}^{n+\frac{1}{2}} + K_2 \delta_y Z_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}^{n+\frac{1}{2}} + K_3 \delta_z Z_{i+\frac{1}{2}, j+\frac{1}{2}, k}^{n+\frac{1}{2}} \\ & = \nabla_Z S_2(Z_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}^{n+\frac{1}{2}})(\dot{\chi})_{i,j,k}^n, \end{aligned} \tag{3.2}$$

with  $M$ ,  $K_p$  ( $p = 1, 2, 3$ ) and  $S_2$  are given by (2.16)–(2.18), and

$$Z_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} = \frac{1}{4}(Z_{i,j+1,k}^{n+1} + Z_{i,j,k}^{n+1} + Z_{i,j+1,k}^n + Z_{i,j,k}^n).$$

Terms  $Z_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}^n$ ,  $Z_{i+1, j+\frac{1}{2}, k+\frac{1}{2}}^{n+\frac{1}{2}}$ ,  $Z_{i, j+\frac{1}{2}, k+\frac{1}{2}}^{n+\frac{1}{2}}$ ,  $Z_{i+\frac{1}{2}, j+1, k+\frac{1}{2}}^{n+\frac{1}{2}}$ ,  $Z_{i+\frac{1}{2}, j, k}^{n+\frac{1}{2}}$  and  $Z_{i+\frac{1}{2}, j+\frac{1}{2}, k}^{n+\frac{1}{2}}$  et al., are defined similarly.

As we stated before  $\dot{\chi}$  may formally be considered as the temporal derivative of the Q-Wiener process, i.e.,  $\dot{\chi} = \frac{dW}{dt}$ . In the numerical experiments in Section 4, we calculate  $(\dot{\chi})_{i,j,k}^n$  as follows

$$(\dot{\chi})_{i,j,k}^n := \frac{(\Delta W)_{i,j,k}^n}{\Delta t} = \frac{W_{i,j,k}^{n+1} - W_{i,j,k}^n}{\Delta t},$$

where  $(\Delta W)_{i,j,k}^n$  means the temporal increment of Wiener process and  $W_{i,j,k}^n$  means  $W(t_n, x_i, y_j, z_k)$ .

This method preserves the following discrete version of stochastic multi-symplectic conservation law; the proof is similar as [2, Theorem 3].

**Theorem 3.1.** *The method (3.2) satisfies the discrete stochastic multi-symplectic conservation law a.s.,*

$$\begin{aligned} & \frac{\omega_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} - \omega_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n}{\Delta t} + \frac{(\kappa_1)_{i+1,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - (\kappa_1)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x} \\ & + \frac{(\kappa_2)_{i+\frac{1}{2},j+1,k+\frac{1}{2}}^{n+\frac{1}{2}} - (\kappa_2)_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y} + \frac{(\kappa_3)_{i+\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}} - (\kappa_3)_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta z} = 0, \end{aligned}$$

where

$$\omega_{i,j,k}^n = dZ_{i,j,k}^{n+1} \wedge MdZ_{i,j,k}^n, \quad (\kappa_p)_{i,j,k}^n = dZ_{i,j,k}^n \wedge K_p dZ_{i,j,k}^n, \quad p = 1, 2, 3.$$

We will present the discrete dissipative property of the discrete energy for Method-I in the following theorem.

**Theorem 3.2.** *Assume that  $\mathbf{E}_{i,j,k}^n$  and  $\mathbf{H}_{i,j,k}^n$  are solutions of numerical method (3.2), then under the periodic boundary condition the discrete energy satisfies the following dissipative property*

$$\Phi^{[I]}(t_{n+1}) = \Phi^{[I]}(t_n) + 2\Delta x \Delta y \Delta z \sum_{i,j,k} \left( \Upsilon_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} (\Delta W)_{i,j,k}^n \right), \tag{3.3}$$

where

$$\Phi^{[I]}(t_n) = \Delta x \Delta y \Delta z \sum_{i,j,k} \left( |\mathbf{E}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n|^2 + |\mathbf{H}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n|^2 \right),$$

and

$$\begin{aligned} \Upsilon_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = & \lambda_2 \left( (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + (H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} \right) \\ & - \lambda_1 \left( (E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + (E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + (E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} \right). \end{aligned} \tag{3.4}$$

**Proof.** We may rewrite method (3.2) into the componentwise form of  $\mathbf{E}$  and  $\mathbf{H}$ ,

$$\delta_t(E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \delta_y(H_3)_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - \delta_z(H_2)_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - \lambda_1(\dot{\chi})_{i,j,k}^n, \tag{3.5a}$$

$$\delta_t(E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \delta_z(H_1)_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - \delta_x(H_3)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \lambda_1(\dot{\chi})_{i,j,k}^n, \tag{3.5b}$$

$$\delta_t(E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \delta_x(H_2)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \delta_y(H_1)_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - \lambda_1(\dot{\chi})_{i,j,k}^n, \tag{3.5c}$$

$$\delta_t(H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \delta_z(E_2)_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - \delta_y(E_3)_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} + \lambda_2(\dot{\chi})_{i,j,k}^n, \tag{3.5d}$$

$$\delta_t(H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \delta_x(E_3)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \delta_z(E_1)_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} + \lambda_2(\dot{\chi})_{i,j,k}^n, \tag{3.5e}$$

$$\delta_t(H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \delta_y(E_1)_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{1}{2}} - \delta_x(E_2)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + \lambda_2(\dot{\chi})_{i,j,k}^n. \tag{3.5f}$$

Multiplying both sides of each equality from (3.5a) to (3.5f) with

$$\begin{aligned} & \Delta t \Delta x \Delta y \Delta z (E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}, \Delta t \Delta x \Delta y \Delta z (E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}, \Delta t \Delta x \Delta y \Delta z (E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}, \\ & \Delta t \Delta x \Delta y \Delta z (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}, \Delta t \Delta x \Delta y \Delta z (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}, \Delta t \Delta x \Delta y \Delta z (H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}, \end{aligned} \tag{3.6}$$

respectively, summing all terms in the above equations over all spatial indices  $i, j, k$ , it yields

$$\Phi^{[I]}(t_{n+1}) = \Phi^{[I]}(t_n) + \mathcal{A} + \mathcal{B}, \tag{3.7}$$

where

$$\begin{aligned} \mathcal{A} = & -2\lambda_1 \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ (E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + (E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + (E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} \right] (\Delta W)_{i,j,k}^n \\ & + 2\lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + (H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} \right] (\Delta W)_{i,j,k}^n. \end{aligned}$$

And  $\mathcal{B}$  represents the corresponding algebraic formula of the first two terms on the right-hand sides of (3.5a) to (3.5f) and the above six terms (3.6). We could show that  $\mathcal{B} = 0$  using the periodic boundary condition. Thus (3.7) leads to the assertion of this theorem.  $\square$

Specially, we could obtain the estimate of the discrete averaged energy in the case that  $W$  only depends on time. The evolution relationship for averaged energy coincides with the continuous case when  $W$  only depends on time.

**Theorem 3.3.** *If  $W = W(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}$  is a Brownian motion, then there exists a constant  $\tilde{K} = 3(\lambda_1^2 + \lambda_2^2)|\Theta|$  such that*

$$\mathcal{E}(\Phi^{[I]}(t_n)) = \mathcal{E}(\Phi^{[I]}(t_0)) + \tilde{K}t_n, \tag{3.8}$$

where  $|\Theta|$  denotes the volume of domain  $\Theta$ .

**Proof.** From the expressions (3.3) and (3.4), we present the analysis of one term as an example, as the other terms can be dealt similarly.

$$\begin{aligned} & 2\lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left( (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} \Delta W_n \right) \\ & = 2\lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left[ (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n + \frac{1}{2} \left( (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} - (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n \right) \right] \Delta W_n. \end{aligned} \tag{3.9}$$

Here  $\Delta W_n = W(t_{n+1}) - W(t_n)$ . Utilizing the properties of the increment of Wiener process leads to

$$\mathcal{E} \left( (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n \Delta W_n \right) = 0.$$

And substituting the equation (3.5d) into  $\left( (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} - (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n \right)$  in (3.9) and using the periodic boundary condition, we obtain

$$2\lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left( (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} \Delta W_n \right) = \lambda_2^2 |\Theta| \Delta t.$$

Similar results hold for others terms. Thus, we get

$$\mathcal{E}(\Phi^{[I]}(t_{n+1})) = \mathcal{E}(\Phi^{[I]}(t_n)) + 3(\lambda_1^2 + \lambda_2^2)|\Theta| \Delta t,$$

which proves the theorem.  $\square$

In order to show that Method-I preserves the discrete version of the averaged divergence, we may need the definition of discrete divergence operator at point  $(x_i, y_j, z_k)$ , which is given as follows; see [13] for the analysis of deterministic case.

$$\bar{\nabla}_{i,j,k}^{[I]} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \delta_x \alpha_{i-\frac{1}{2},j\pm\frac{1}{2},k\pm\frac{1}{2}} + \delta_y \beta_{i\pm\frac{1}{2},j-\frac{1}{2},k\pm\frac{1}{2}} + \delta_z \gamma_{i\pm\frac{1}{2},j\pm\frac{1}{2},k-\frac{1}{2}}, \tag{3.10}$$

where

$$\begin{aligned} \alpha_{i-\frac{1}{2},j\pm\frac{1}{2},k\pm\frac{1}{2}} & := \alpha_{i-\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + \alpha_{i-\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} + \alpha_{i-\frac{1}{2},j-\frac{1}{2},k+\frac{1}{2}} + \alpha_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}}, \\ \beta_{i\pm\frac{1}{2},j-\frac{1}{2},k\pm\frac{1}{2}} & := \beta_{i+\frac{1}{2},j-\frac{1}{2},k+\frac{1}{2}} + \beta_{i-\frac{1}{2},j-\frac{1}{2},k+\frac{1}{2}} + \beta_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}} + \beta_{i+\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}}, \\ \gamma_{i\pm\frac{1}{2},j\pm\frac{1}{2},k-\frac{1}{2}} & := \gamma_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} + \gamma_{i+\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}} + \gamma_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}} + \gamma_{i-\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}. \end{aligned}$$

The following theorem shows that Method-I preserves the discrete version of averaged divergence.



**Theorem 3.4.** The numerical discretization (3.5) to equations (2.1) preserves the following discrete averaged divergence, i.e.,

$$\mathcal{E} \left( \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{E}^{n+1} \right) = \mathcal{E} \left( \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{E}^n \right), \quad \mathcal{E} \left( \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{H}^{n+1} \right) = \mathcal{E} \left( \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{H}^n \right). \quad (3.11)$$

**Proof.** The proof of the two assertions are similar, so here we only present that for electric field  $\mathbf{E}$ . By the definition (3.10), we have

$$\bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{E}^{n+1} - \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{E}^n = \Delta t \left[ \delta_x \delta_t (E_1)_{i-\frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2}}^n + \delta_y \delta_t (E_2)_{i \pm \frac{1}{2}, j-\frac{1}{2}, k \pm \frac{1}{2}}^n + \delta_z \delta_t (E_3)_{i \pm \frac{1}{2}, j \pm \frac{1}{2}, k-\frac{1}{2}}^n \right]$$

Utilizing the method (3.5a)–(3.5c) to replace the temporal difference expressions of  $E_1$ ,  $E_2$  and  $E_3$  in the above equation leads to

$$\begin{aligned} \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{E}^{n+1} - \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{E}^n &= \Delta t \delta_x \left[ \delta_y (H_3)_{i-\frac{1}{2}, (j-\frac{1}{2}) \pm \frac{1}{2}, k \pm \frac{1}{2}}^{n+\frac{1}{2}} - \delta_z (H_2)_{i-\frac{1}{2}, j \pm \frac{1}{2}, (k-\frac{1}{2}) \pm \frac{1}{2}}^{n+\frac{1}{2}} \right] \\ &\quad + \Delta t \delta_y \left[ \delta_z (H_1)_{i \pm \frac{1}{2}, j-\frac{1}{2}, (k-\frac{1}{2}) \pm \frac{1}{2}}^{n+\frac{1}{2}} - \delta_x (H_3)_{(i-\frac{1}{2}) \pm \frac{1}{2}, j-\frac{1}{2}, k \pm \frac{1}{2}}^{n+\frac{1}{2}} \right] \\ &\quad + \Delta t \delta_z \left[ \delta_x (H_2)_{(i-\frac{1}{2}) \pm \frac{1}{2}, j \pm \frac{1}{2}, k-\frac{1}{2}}^{n+\frac{1}{2}} - \delta_y (H_1)_{i \pm \frac{1}{2}, (j-\frac{1}{2}) \pm \frac{1}{2}, k-\frac{1}{2}}^{n+\frac{1}{2}} \right] \\ &\quad + \text{term(b)}, \end{aligned}$$

where

$$\begin{aligned} \text{term(b)} &= -\lambda_1 \delta_x (\Delta W)_{i-1, (j-\frac{1}{2}) \pm \frac{1}{2}, (k-\frac{1}{2}) \pm \frac{1}{2}}^n - \lambda_1 \delta_y (\Delta W)_{(i-\frac{1}{2}) \pm \frac{1}{2}, j-1, (k-\frac{1}{2}) \pm \frac{1}{2}}^n \\ &\quad - \lambda_1 \delta_z (\Delta W)_{(i-\frac{1}{2}) \pm \frac{1}{2}, (j-\frac{1}{2}) \pm \frac{1}{2}, k-1}^n, \end{aligned}$$

with  $(\Delta W)_{i-1, (j-\frac{1}{2}) \pm \frac{1}{2}, (k-\frac{1}{2}) \pm \frac{1}{2}}^n = (\Delta W)_{i-1, j-1, k}^n + (\Delta W)_{i-1, j, k-1}^n + (\Delta W)_{i-1, j-1, k-1}^n + (\Delta W)_{i-1, j, k}^n$  and other terms being defined in the same way. Utilizing similar approach as in deterministic case (see [13]), we could show that the left hand side of the above formula equals to term(b). By the property of Wiener process, we have

$$\mathcal{E} \left( \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{E}^{n+1} - \bar{\nabla}_{i,j,k}^{[I]} \cdot \mathbf{E}^n \right) = \mathcal{E} \left( \text{term(b)} \right) = 0. \quad (3.12)$$

Thus the proof is completed.  $\square$

### 3.2. Method-II

As is well known, for the numerical simulation of deterministic Maxwell equations, Yee’s method is the basis of the highly popular CEM numerical methods known as the finite-difference time-domain (FDTD) methods (see the original work [15]). It is constructed by central difference in both spatial and temporal directions based on a half-step staggered grid. With the difference operators defined in (3.1), we generalize an equivalent form of Yee’s method [8] to discretize the stochastic Maxwell equations (2.1) as follows:

$$\bar{\delta}_t (E_1)_{i,j,k}^n = \bar{\delta}_y (H_3)_{i,j,k}^n - \bar{\delta}_z (H_2)_{i,j,k}^n - \lambda_1 (\tilde{\chi})_{i,j,k}^{n+1}, \quad (3.13a)$$

$$\bar{\delta}_t (E_2)_{i,j,k}^n = \bar{\delta}_z (H_1)_{i,j,k}^n - \bar{\delta}_x (H_3)_{i,j,k}^n - \lambda_1 (\tilde{\chi})_{i,j,k}^{n+1}, \quad (3.13b)$$

$$\bar{\delta}_t (E_3)_{i,j,k}^n = \bar{\delta}_x (H_2)_{i,j,k}^n - \bar{\delta}_y (H_1)_{i,j,k}^n - \lambda_1 (\tilde{\chi})_{i,j,k}^{n+1}, \quad (3.13c)$$

$$\bar{\delta}_t (H_1)_{i,j,k}^n = -\bar{\delta}_y (E_3)_{i,j,k}^n + \bar{\delta}_z (E_2)_{i,j,k}^n + \lambda_2 (\tilde{\chi})_{i,j,k}^{n+1}, \quad (3.13d)$$

$$\bar{\delta}_t (H_2)_{i,j,k}^n = -\bar{\delta}_z (E_1)_{i,j,k}^n + \bar{\delta}_x (E_3)_{i,j,k}^n + \lambda_2 (\tilde{\chi})_{i,j,k}^{n+1}, \quad (3.13e)$$

$$\bar{\delta}_t (H_3)_{i,j,k}^n = -\bar{\delta}_x (E_2)_{i,j,k}^n + \bar{\delta}_y (E_1)_{i,j,k}^n + \lambda_2 (\tilde{\chi})_{i,j,k}^{n+1}, \quad (3.13f)$$

where

$$(\tilde{\chi})_{i,j,k}^{n+1} = \frac{W_{i,j,k}^{n+1} - W_{i,j,k}^{n-1}}{2\Delta t}.$$

Clearly, the above method conserves the following discrete version of stochastic multi-symplectic conservation law.

**Theorem 3.5.** Method (3.13) possesses the discrete stochastic multi-symplectic conservation law a.s.,

$$\frac{\omega_{i,j,k}^{n+\frac{1}{2}} - \omega_{i,j,k}^{n-\frac{1}{2}}}{\Delta t} + \frac{(\kappa_1)_{i+\frac{1}{2},j,k}^n - (\kappa_1)_{i-\frac{1}{2},j,k}^n}{\Delta x} + \frac{(\kappa_2)_{i,j+\frac{1}{2},k}^n - (\kappa_2)_{i,j-\frac{1}{2},k}^n}{\Delta y} + \frac{(\kappa_3)_{i,j,k+\frac{1}{2}}^n - (\kappa_3)_{i,j,k-\frac{1}{2}}^n}{\Delta z} = 0,$$

where

$$\begin{aligned} \omega_{i,j,k}^{n+\frac{1}{2}} &= dZ_{i,j,k}^n \wedge MdZ_{i,j,k}^{n+1}, & (\kappa_1)_{i+\frac{1}{2},j,k}^n &= dZ_{i,j,k}^n \wedge K_1 dZ_{i+1,j,k}^n, \\ (\kappa_2)_{i,j+\frac{1}{2},k}^n &= dZ_{i,j,k}^n \wedge K_2 dZ_{i,j+1,k}^n, & (\kappa_3)_{i,j,k+\frac{1}{2}}^n &= dZ_{i,j,k}^n \wedge K_3 dZ_{i,j,k+1}^n. \end{aligned}$$

Also, we will consider the properties of the discrete averaged energy and the discrete averaged divergence in the following contents.

**Theorem 3.6.** Assume that  $\mathbf{E}_{i,j,k}^n$  and  $\mathbf{H}_{i,j,k}^n$  are solutions of numerical method (3.13), then under the periodic boundary condition the discrete energy satisfies

$$\Phi^{[M]}(t_{n+1}) = \Phi^{[M]}(t_n) + \Delta x \Delta y \Delta z \sum_{i,j,k} \left( \Upsilon_{i,j,k}^n (W_{i,j,k}^{n+1} - W_{i,j,k}^{n-1}) \right), \tag{3.14}$$

where

$$\Phi^{[M]}(t_{n+1}) = \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \langle \mathbf{E}_{i,j,k}^{n+1}, \mathbf{E}_{i,j,k}^n \rangle + \langle \mathbf{H}_{i,j,k}^{n+1}, \mathbf{H}_{i,j,k}^n \rangle \right]$$

and

$$\Upsilon_{i,j,k}^n = \lambda_2 \left( (H_1)_{i,j,k}^n + (H_2)_{i,j,k}^n + (H_3)_{i,j,k}^n \right) - \lambda_1 \left( (E_1)_{i,j,k}^n + (E_2)_{i,j,k}^n + (E_3)_{i,j,k}^n \right).$$

**Proof.** Multiplying both sides of each equation from (3.13a) to (3.13f) with

$$\begin{aligned} &2\Delta t \Delta x \Delta y \Delta z (E_1)_{i,j,k}^n, \quad 2\Delta t \Delta x \Delta y \Delta z (E_2)_{i,j,k}^n, \quad 2\Delta t \Delta x \Delta y \Delta z (E_3)_{i,j,k}^n, \\ &2\Delta t \Delta x \Delta y \Delta z (H_1)_{i,j,k}^n, \quad 2\Delta t \Delta x \Delta y \Delta z (H_2)_{i,j,k}^n, \quad 2\Delta t \Delta x \Delta y \Delta z (H_3)_{i,j,k}^n, \end{aligned}$$

respectively. The proof is finished by summing all terms in over all spatial indices  $i, j, k$  together, and using the periodic boundary condition. □

Moreover, we have the following estimation for the discrete averaged energy.

**Theorem 3.7.** There exists a constant  $\hat{K} = 3(\lambda_1^2 + \lambda_2^2) \bar{V}^Q(\Theta)$  such that

$$\mathcal{E} \left( \Phi^{[M]}(t_n) \right) = \mathcal{E} \left( \Phi^{[M]}(t_0) \right) + \hat{K} t_n. \tag{3.15}$$

**Proof.** We need to estimate each term in  $\Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left( \Upsilon_{i,j,k}^n (W_{i,j,k}^{n+1} - W_{i,j,k}^{n-1}) \right)$ . For the first term we have

$$\begin{aligned} &\lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left[ (H_1)_{i,j,k}^n (W_{i,j,k}^{n+1} - W_{i,j,k}^{n-1}) \right] = \lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left[ (H_1)_{i,j,k}^n (\Delta W)_{i,j,k}^{n-1} \right] \\ &= \lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left[ \left( (H_1)_{i,j,k}^n - (H_1)_{i,j,k}^{n-2} \right) (\Delta W)_{i,j,k}^{n-1} \right] \\ &= \lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left\{ \left[ -\Delta t \bar{\delta}_y (E_3)_{i,j,k}^{n-1} + \Delta t \bar{\delta}_z (E_2)_{i,j,k}^{n-1} + \lambda_2 (W_{i,j,k}^n - W_{i,j,k}^{n-2}) \right] (\Delta W)_{i,j,k}^{n-1} \right\} \\ &= \lambda_2^2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left\{ (W_{i,j,k}^n - W_{i,j,k}^{n-2}) (W_{i,j,k}^n - W_{i,j,k}^{n-1}) \right\} \\ &= \lambda_2^2 \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E} \left\{ (W_{i,j,k}^n - W_{i,j,k}^{n-1})^2 \right\} = \lambda_2^2 \bar{V}^Q(\Theta) \Delta t \end{aligned}$$

where  $\bar{V}^Q(\Theta) = \Delta x \Delta y \Delta z \sum_{i,j,k} \sum_m \eta_m e_m^2(x_i, y_j, z_k)$ . Here we mainly use the independent properties of Wiener increments. Because other terms could be estimated similarly, we finish the proof by noting that  $\hat{K} = 3(\lambda_1^2 + \lambda_2^2)\bar{V}^Q(\Theta)$ .  $\square$

Note that the constant  $\bar{V}^Q(\Theta)$  here may be regarded as the approximation of  $Tr(Q)$ , i.e.,

$$\bar{V}^Q(\Theta) \approx \sum_m \eta_m \int_{\Theta} e_m^2(x, y, z) dx dy dz = \sum_m \eta_m = Tr(Q).$$

Furthermore, the method (3.13) preserves the following discrete averaged divergence.

**Theorem 3.8.** *The method (3.13) preserves the following discrete averaged divergence*

$$\begin{aligned} \mathcal{E}\left(\bar{\nabla}^{[III]} \cdot \mathbf{E}_{i,j,k}^{n+\frac{1}{2}}\right) &= \mathcal{E}\left(\bar{\nabla}^{[III]} \cdot \mathbf{E}_{i,j,k}^{n-\frac{1}{2}}\right), \\ \mathcal{E}\left(\bar{\nabla}^{[III]} \cdot \mathbf{H}_{i,j,k}^{n+\frac{1}{2}}\right) &= \mathcal{E}\left(\bar{\nabla}^{[III]} \cdot \mathbf{H}_{i,j,k}^{n-\frac{1}{2}}\right), \end{aligned} \tag{3.16}$$

where  $\bar{\nabla}^{[III]} = (\bar{\delta}_x, \bar{\delta}_y, \bar{\delta}_z)^T$ .

The proof of this theorem is similar to that of Theorem 3.4, so we omit it here.

### 3.3. Method-III

We use the central finite difference in spatial direction and implicit midpoint method in temporal direction, then we refer to this particular discretization as Method-III (see [13] for deterministic case)

$$M\delta_t Z_{i,j,k}^n + K_1 \bar{\delta}_x Z_{i,j,k}^{n+\frac{1}{2}} + K_2 \bar{\delta}_y Z_{i,j,k}^{n+\frac{1}{2}} + K_3 \bar{\delta}_z Z_{i,j,k}^{n+\frac{1}{2}} = \nabla_z S_2(Z_{i,j,k}^{n+\frac{1}{2}}) (\dot{X})_{i,j,k}^n. \tag{3.17}$$

It is shown that method (3.17) preserves the stochastic multi-symplectic conservation law.

**Theorem 3.9.** *The method (3.17) satisfies the discrete stochastic multi-symplectic conservation law a.s.,*

$$\frac{\omega_{i,j,k}^{n+1} - \omega_{i,j,k}^n}{\Delta t} + \frac{(\kappa_1)_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} - (\kappa_1)_{i-\frac{1}{2},j,k}^{n+\frac{1}{2}}}{\Delta x} + \frac{(\kappa_2)_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} - (\kappa_2)_{i,j-\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta y} + \frac{(\kappa_3)_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}} - (\kappa_3)_{i,j,k-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z} = 0, \tag{3.18}$$

where

$$\begin{aligned} \omega_{i,j,k}^{n+1} &= dZ_{i,j,k}^{n+1} \wedge MdZ_{i,j,k}^{n+1}, & (\kappa_1)_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} &= dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_1 dZ_{i+1,j,k}^{n+\frac{1}{2}}, \\ (\kappa_2)_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} &= dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_2 dZ_{i,j+1,k}^{n+\frac{1}{2}}, & (\kappa_3)_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}} &= dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_3 dZ_{i,j,k+1}^{n+\frac{1}{2}}. \end{aligned}$$

**Proof.** We take differential in the phase space on both sides of (3.17) to obtain

$$\begin{aligned} &2\Delta x \Delta y \Delta z M(dZ_{i,j,k}^{n+1} - dZ_{i,j,k}^n) + \Delta t \Delta y \Delta z K_1(dZ_{i+1,j,k}^{n+\frac{1}{2}} - dZ_{i-1,j,k}^{n+\frac{1}{2}}) \\ &+ \Delta t \Delta x \Delta z K_2(dZ_{i,j+1,k}^{n+\frac{1}{2}} - dZ_{i,j-1,k}^{n+\frac{1}{2}}) + \Delta t \Delta x \Delta y K_3(dZ_{i,j,k+1}^{n+\frac{1}{2}} - dZ_{i,j,k-1}^{n+\frac{1}{2}}) \\ &= 2\Delta x \Delta y \Delta z \nabla^2 S_2(Z_{i,j,k}^{n+\frac{1}{2}}) dZ_{i,j,k}^{n+\frac{1}{2}} (\Delta W)_{i,j,k}^n. \end{aligned}$$

Then taking  $dZ_{i,j,k}^{n+\frac{1}{2}} = \frac{1}{2}(dZ_{i,j,k}^{n+1} + dZ_{i,j,k}^n)$  and performing wedge product with the above equation yields

$$\begin{aligned} &\Delta x \Delta y \Delta z (dZ_{i,j,k}^{n+1} \wedge MdZ_{i,j,k}^{n+1} - dZ_{i,j,k}^n \wedge MdZ_{i,j,k}^n) \\ &+ \Delta t \Delta y \Delta z (dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_1 dZ_{i+1,j,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_1 dZ_{i-1,j,k}^{n+\frac{1}{2}}) \\ &+ \Delta t \Delta x \Delta z (dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_2 dZ_{i,j+1,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_2 dZ_{i,j-1,k}^{n+\frac{1}{2}}) \\ &+ \Delta t \Delta x \Delta y (dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_3 dZ_{i,j,k+1}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_3 dZ_{i,j,k-1}^{n+\frac{1}{2}}) \\ &= 0. \end{aligned}$$

Thus we finish the proof by denoting the definitions of discrete differential 2-forms.  $\square$

We also rewrite (3.17) into the component form of  $\mathbf{E}$  and  $\mathbf{H}$  as follows:

$$\delta_t(E_1)_{i,j,k}^n = \bar{\delta}_y(H_3)_{i,j,k}^{n+\frac{1}{2}} - \bar{\delta}_z(H_2)_{i,j,k}^{n+\frac{1}{2}} - \lambda_1(\dot{\chi})_{i,j,k}^n, \tag{3.19a}$$

$$\delta_t(E_2)_{i,j,k}^n = \bar{\delta}_z(H_1)_{i,j,k}^{n+\frac{1}{2}} - \bar{\delta}_x(H_3)_{i,j,k}^{n+\frac{1}{2}} - \lambda_1(\dot{\chi})_{i,j,k}^n, \tag{3.19b}$$

$$\delta_t(E_3)_{i,j,k}^n = \bar{\delta}_x(H_2)_{i,j,k}^{n+\frac{1}{2}} - \bar{\delta}_y(H_1)_{i,j,k}^{n+\frac{1}{2}} - \lambda_1(\dot{\chi})_{i,j,k}^n, \tag{3.19c}$$

$$\delta_t(H_1)_{i,j,k}^n = -\bar{\delta}_y(E_3)_{i,j,k}^{n+\frac{1}{2}} + \bar{\delta}_z(E_2)_{i,j,k}^{n+\frac{1}{2}} + \lambda_2(\dot{\chi})_{i,j,k}^n, \tag{3.19d}$$

$$\delta_t(H_2)_{i,j,k}^n = -\bar{\delta}_z(E_1)_{i,j,k}^{n+\frac{1}{2}} + \bar{\delta}_x(E_3)_{i,j,k}^{n+\frac{1}{2}} + \lambda_2(\dot{\chi})_{i,j,k}^n, \tag{3.19e}$$

$$\delta_t(H_3)_{i,j,k}^n = -\bar{\delta}_x(E_2)_{i,j,k}^{n+\frac{1}{2}} + \bar{\delta}_y(E_1)_{i,j,k}^{n+\frac{1}{2}} + \lambda_2(\dot{\chi})_{i,j,k}^n. \tag{3.19f}$$

The following theorem states the dissipative property for the discrete energy of Method-III.

**Theorem 3.10.** Assume that  $\mathbf{E}_{i,j,k}^n$  and  $\mathbf{H}_{i,j,k}^n$  are solutions of numerical method (3.19), then under the periodic boundary condition, the discrete energy satisfies the following dissipative property

$$\Phi^{[III]}(t_{n+1}) = \Phi^{[III]}(t_n) + 2\Delta x \Delta y \Delta z \sum_{i,j,k} \left( \Upsilon_{i,j,k}^{n+\frac{1}{2}} (\Delta W)_{i,j,k}^n \right), \tag{3.20}$$

where

$$\Phi^{[III]}(t_n) = \Delta x \Delta y \Delta z \sum_{i,j,k} (|\mathbf{E}_{i,j,k}^n|^2 + |\mathbf{H}_{i,j,k}^n|^2),$$

and

$$\Upsilon_{i,j,k}^{n+\frac{1}{2}} = \lambda_2 \left( (H_1)_{i,j,k}^{n+\frac{1}{2}} + (H_2)_{i,j,k}^{n+\frac{1}{2}} + (H_3)_{i,j,k}^{n+\frac{1}{2}} \right) - \lambda_1 \left( (E_1)_{i,j,k}^{n+\frac{1}{2}} + (E_2)_{i,j,k}^{n+\frac{1}{2}} + (E_3)_{i,j,k}^{n+\frac{1}{2}} \right). \tag{3.21}$$

The proof is similar to that of Theorem 3.2, so we omit it here.

In the following theorem we give an estimation about the evolution of the discrete averaged energy.

**Theorem 3.11.** There exists a constant  $\bar{K} > 0$  such that

$$\mathcal{E} \left( \Phi^{[III]}(t_n) \right) - \mathcal{E} \left( \Phi^{[III]}(t_0) \right) \leq \bar{K} t_n. \tag{3.22}$$

**Proof.** The estimate of each term in the second term on the right-hand side of (3.20) is similar, so here we present estimates of terms related with  $H_1$  and  $E_1$  as examples.

Using the identity  $2a^{n+\frac{1}{2}} = 2a^n + (a^{n+1} - a^n)$ , the independent property of Wiener increment and (3.19d), we get

$$\begin{aligned} & \mathcal{E} \left\{ 2\lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} (H_1)_{i,j,k}^{n+\frac{1}{2}} (\Delta W)_{i,j,k}^n \right\} \\ &= \mathcal{E} \left\{ \lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ ((H_1)_{i,j,k}^{n+1} - (H_1)_{i,j,k}^n) (\Delta W)_{i,j,k}^n \right] \right\} \\ &= \mathcal{E} \left\{ \lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \frac{\Delta t}{2\Delta z} \left( (E_2)_{i,j,k+1}^{n+\frac{1}{2}} - (E_2)_{i,j,k-1}^{n+\frac{1}{2}} \right) \right. \right. \\ &\quad \left. \left. - \frac{\Delta t}{2\Delta y} \left( (E_3)_{i,j+1,k}^{n+\frac{1}{2}} - (E_3)_{i,j-1,k}^{n+\frac{1}{2}} \right) + \lambda_2 (\Delta W)_{i,j,k}^n \right] (\Delta W)_{i,j,k}^n \right\} \\ &= \mathcal{E} \left\{ \lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ -\Delta t (E_2)_{i,j,k}^{n+\frac{1}{2}} \bar{\delta}_z (\Delta W)_{i,j,k}^n + \Delta t (E_3)_{i,j,k}^{n+\frac{1}{2}} \bar{\delta}_y (\Delta W)_{i,j,k}^n + \lambda_2 ((\Delta W)_{i,j,k}^n)^2 \right] \right\}, \end{aligned}$$

where we are benefit from the periodic boundary condition. By Young's inequality, we may obtain

$$\begin{aligned}
& \mathcal{E} \left\{ 2\lambda_2 \Delta x \Delta y \Delta z \sum_{i,j,k} (H_1)_{i,j,k}^{n+\frac{1}{2}} (\Delta W)_{i,j,k}^n \right\} \\
& \leq \frac{1}{6} \Delta t^2 \Delta x \Delta y \Delta z \mathcal{E} \left[ \sum_{ijk} ((E_3)_{i,j,k}^{n+\frac{1}{2}})^2 + \sum_{ijk} ((E_2)_{i,j,k}^{n+\frac{1}{2}})^2 \right] \\
& \quad + \frac{3}{2} \lambda_2^2 \Delta t \Delta x \Delta y \Delta z \sum_{ijk} \sum_m \eta_m (\bar{\delta}_z e_m(x_i, y_j, z_k))^2 \\
& \quad + \frac{3}{2} \lambda_2^2 \Delta t \Delta x \Delta y \Delta z \sum_{ijk} \sum_m \eta_m (\bar{\delta}_y e_m(x_i, y_j, z_k))^2 + \lambda_2^2 \bar{V}^Q(\Theta) \Delta t, \tag{3.23}
\end{aligned}$$

where  $\bar{V}^Q(\Theta) := \Delta x \Delta y \Delta z \sum_{i,j,k} \sum_m \eta_m e_m^2(x_i, y_j, z_k)$ . Similarly, for term related with  $E_1$ , we have

$$\begin{aligned}
& \mathcal{E} \left\{ -2\lambda_1 \Delta x \Delta y \Delta z \sum_{i,j,k} (E_1)_{i,j,k}^{n+\frac{1}{2}} (\Delta W)_{i,j,k}^n \right\} \\
& = \mathcal{E} \left\{ -\lambda_1 \Delta x \Delta y \Delta z \Delta t \sum_{i,j,k} \left[ -(H_3)_{i,j,k}^{n+\frac{1}{2}} \bar{\delta}_y (\Delta W)_{i,j,k}^n + (H_2)_{i,j,k}^{n+\frac{1}{2}} \bar{\delta}_z (\Delta W)_{i,j,k}^n - \lambda_1 ((\Delta W)_{i,j,k}^n)^2 \right] \right\} \\
& \leq \frac{1}{6} \Delta t^2 \Delta x \Delta y \Delta z \mathcal{E} \left[ \sum_{ijk} ((H_3)_{i,j,k}^{n+\frac{1}{2}})^2 + \sum_{ijk} ((H_2)_{i,j,k}^{n+\frac{1}{2}})^2 \right] \\
& \quad + \frac{3}{2} \lambda_1^2 \Delta t \Delta x \Delta y \Delta z \sum_{ijk} \sum_m \eta_m (\bar{\delta}_y e_m(x_i, y_j, z_k))^2 \\
& \quad + \frac{3}{2} \lambda_1^2 \Delta t \Delta x \Delta y \Delta z \sum_{ijk} \sum_m \eta_m (\bar{\delta}_z e_m(x_i, y_j, z_k))^2 + \lambda_1^2 \bar{V}^Q(\Theta) \Delta t. \tag{3.24}
\end{aligned}$$

Therefore, by denoting

$$\hat{V}^Q(\Theta) := \bar{V}^Q(\Theta) + \Delta x \Delta y \Delta z \sum_{ijk} \sum_m \eta_m \left( (\bar{\delta}_x e_m(x_i, y_j, z_k))^2 + (\bar{\delta}_y e_m(x_i, y_j, z_k))^2 + (\bar{\delta}_z e_m(x_i, y_j, z_k))^2 \right),$$

we have

$$\mathcal{E}(\Phi^{[III]}(t_{n+1})) \leq \mathcal{E}(\Phi^{[III]}(t_n)) + \frac{1}{6} \Delta t^2 \mathcal{E}(\Phi^{[III]}(t_{n+1})) + 3(\lambda_1^2 + \lambda_2^2) \hat{V}^Q(\Theta) \Delta t.$$

By Gronwall inequality, there exist constants  $\Delta t^*$  and  $K := K(\hat{V}^Q(\Theta), \lambda_1, \lambda_2, T)$  such that for  $\Delta t \leq \Delta t^*$ , we have  $\mathcal{E}(\Phi^{[III]}(t_n)) \leq K, \forall n = 0, 1, \dots, N$ . Combing this boundedness together with (3.25), there exists another constant  $\bar{K} := \bar{K}(\hat{V}^Q(\Theta), \lambda_1, \lambda_2, T)$  such that

$$\mathcal{E}(\Phi^{[III]}(t_n)) - \mathcal{E}(\Phi^{[III]}(t_0)) \leq \bar{K} t_n.$$

Thus the proof is finished.  $\square$

Note that the notation  $\hat{V}^Q(\Theta)$  is an approximation of  $\|Q^{\frac{1}{2}}\|_{HS(L^2(\Theta), H^1(\Theta))}^2$ , i.e.,

$$\hat{V}^Q(\Theta) \approx \sum_m \eta_m \int_{\Theta} (|e_m(x, y, z)|^2 + |\nabla e_m(x, y, z)|^2) dx dy dz = \|Q^{\frac{1}{2}}\|_{HS(L^2(\Theta), H^1(\Theta))}^2,$$

while  $\bar{V}^Q(\Theta) \approx \sum_m \eta_m = \text{Tr}(Q) = \|Q^{\frac{1}{2}}\|_{HS(L^2(\Theta), L^2(\Theta))}^2$ .

**Remark 2.** If  $W = W(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}$  is a Brownian motion, the same as Theorem 3.3, we have

$$\mathcal{E}(\Phi^{[III]}(t_n)) = \mathcal{E}(\Phi^{[III]}(t_0)) + \bar{K} t_n,$$

with  $\bar{K} = 3(\lambda_1^2 + \lambda_2^2)|\Theta|$ .

Define  $\bar{\nabla}^{[III]} = (\bar{\delta}_x, \bar{\delta}_y, \bar{\delta}_z)^T$ , then Method-III can preserve the following discrete averaged divergence. The proof is similar to that of [Theorem 3.4](#).

**Theorem 3.12.** *The numerical discretization (3.19) to stochastic Maxwell equations (2.1) preserves the following discrete averaged divergence*

$$\begin{aligned} \mathcal{E} \left( \bar{\nabla}^{[III]} \cdot \mathbf{E}_{i,j,k}^{n+1} \right) &= \mathcal{E} \left( \bar{\nabla}^{[III]} \cdot \mathbf{E}_{i,j,k}^n \right), \\ \mathcal{E} \left( \bar{\nabla}^{[III]} \cdot \mathbf{H}_{i,j,k}^{n+1} \right) &= \mathcal{E} \left( \bar{\nabla}^{[III]} \cdot \mathbf{H}_{i,j,k}^n \right). \end{aligned} \tag{3.25}$$

We may conclude that all of the three numerical methods are shown to be stochastic multi-symplectic and preserve the conservation law of the corresponding version of discrete averaged divergence. For the continuous problem, we prove that the averaged energy evolves linearly with respect to time, while each method in our consideration preserves this property to certain level. We show that this linear growth property is preserved well by Method-II, whereas Method-I and Method-III conserve this property in the case that the noise only depends on temporal variable. Moreover, we could prove that for space–time noise, the corresponding discrete averaged energy of Method-III grows at most linearly.

#### 4. Numerical results

In this section, we mainly focus on the simulation of 2-D stochastic Maxwell equations with additive noise, for which the electric field and the magnetic field are  $\mathbf{E} = (0, 0, E_3)^T$ ,  $\mathbf{H} = (H_1, H_2, 0)^T$ , respectively. I.e.,

$$\begin{cases} \frac{\partial E_3}{\partial t} = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} - \lambda_1 \dot{\chi} & \text{in } (0, T) \times \Theta, \\ \frac{\partial H_1}{\partial t} = -\frac{\partial E_3}{\partial y} + \lambda_2 \dot{\chi} & \text{in } (0, T) \times \Theta, \\ \frac{\partial H_2}{\partial t} = \frac{\partial E_3}{\partial x} + \lambda_2 \dot{\chi} & \text{in } (0, T) \times \Theta, \end{cases} \tag{4.1}$$

with  $\Theta = [0, \frac{2}{3}] \times [0, \frac{1}{2}]$ ,  $T = 1$  and initial data being

$$\begin{aligned} E_3(x, y, 0) &= \sin(3\pi x) \sin(4\pi y), \\ H_1(x, y, 0) &= -\frac{4}{5} \cos(3\pi x) \cos(4\pi y), \\ H_2(x, y, 0) &= -\frac{3}{5} \sin(3\pi x) \sin(4\pi y). \end{aligned}$$

Hereafter, we choose the values of  $\{e_{m,\ell}(x, y)\}_{m,\ell \in \mathbb{N}}$  and  $\{\eta_{m,\ell}\}_{m,\ell \in \mathbb{N}}$  as

$$e_{m,\ell}(x, y) = 2\sqrt{3} \sin\left(\frac{3}{2}m\pi x\right) \sin(2\ell\pi y), \quad \eta_{m,\ell} = \frac{1}{m^3 + \ell^3}. \tag{4.2}$$

By the definition of Wiener process (2.4), we have

$$(\Delta W)_{i,j}^n := W_{i,j}^{n+1} - W_{i,j}^n = \sum_{m,\ell=1}^{\infty} 2\sqrt{\frac{3}{m^3 + \ell^3}} \sin\left(\frac{3}{2}m\pi x_i\right) \sin(2\ell\pi y_j) \sqrt{\Delta t} \xi_{m,\ell}^n, \tag{4.3}$$

with  $\{\xi_{m,\ell}^n\}$  being independent  $N(0, 1)$ -random variables.

In the performance of numerical methods, it is necessary to truncate this infinity sum. [Fig. 4.1](#) displays the value of  $a(m, \ell) = 2\sqrt{\frac{3}{m^3 + \ell^3}}$  with respect to  $m$  and  $\ell$ . Observe that, after  $m, \ell$  larger than 25, the values of  $a(m, \ell)$  tend to zero. Thus we truncate the noise by taking the sum of 50 terms for both parameters  $m$  and  $\ell$  in the following experiments.

And we take the temporal step-size  $\Delta t = 0.001$  and the spatial mesh grid size  $\Delta x = \Delta y = \frac{1}{150}$ . In order to show the influence of noise on the solution, we scale the values of  $\lambda_1 = \lambda_2 = \lambda$  by  $\lambda = 0$ ,  $\lambda = 0.01$ ,  $\lambda = 0.05$  and  $\lambda = 0.1$ , respectively. Taking the magnetic field  $H_1$  for an example, [Fig. 4.2](#) shows the contours until  $t = T$ , by using Method-I corresponding to different scales of the noise. We observe that the perturbation of magnetic wave  $H_1$  becomes much more obvious both in  $x$  and  $y$  directions due to the increase of the scale of the noise.

Next, we focus on numerically performing the dissipative properties of averaged energy. Based on [Theorems 3.2, 3.6 and 3.10](#) for three numerical methods applied to 3-D stochastic Maxwell equations, we present the concrete form for 2-D case (4.1) respectively.

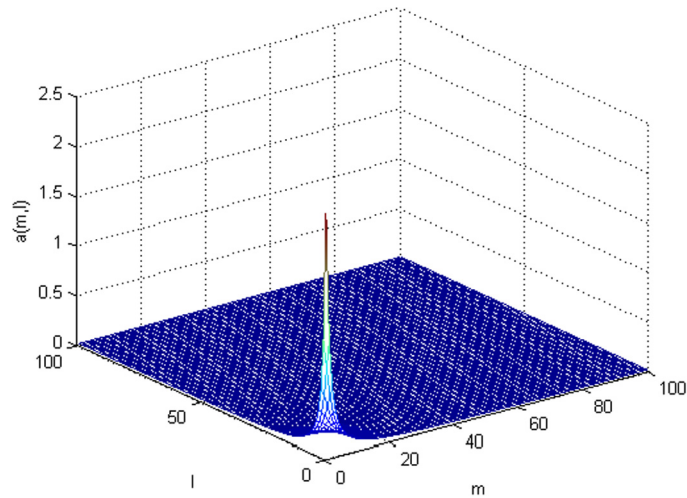


Fig. 4.1. The value of  $a(m, l)$  with respect to  $m$  and  $l$ .

(1) Method-I

$$\Phi^{[I]}(t_{n+1}) = \Phi^{[I]}(t_n) + 2\Delta x \Delta y \sum_{i,j} \left( \Upsilon_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+\frac{1}{2}} (\Delta W)_{i,j}^n \right),$$

where

$$\Phi^{[I]}(t_{n+1}) = \Delta x \Delta y \sum_{i,j} \left[ \left| (E_3)_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+1} \right|^2 + \left| (H_1)_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+1} \right|^2 + \left| (H_2)_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+1} \right|^2 \right]$$

and

$$\Upsilon_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+\frac{1}{2}} = \lambda_2 \left( (H_1)_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+\frac{1}{2}} + (H_2)_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+\frac{1}{2}} \right) - \lambda_1 (E_3)_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+\frac{1}{2}},$$

(2) Method-II

$$\Phi^{[II]}(t_{n+1}) = \Phi^{[II]}(t_n) + \Delta x \Delta y \sum_{i,j} \left( \Upsilon_{i,j}^n (W_{i,j}^{n+1} - W_{i,j}^{n-1}) \right),$$

where

$$\Phi^{[II]}(t_{n+1}) = \Delta x \Delta y \sum_{i,j} \left( (E_3)_{i,j}^{n+1} (E_3)_{i,j}^n + (H_1)_{i,j}^{n+1} (H_1)_{i,j}^n + (H_2)_{i,j}^{n+1} (H_2)_{i,j}^n \right)$$

and

$$\Upsilon_{i,j}^n = \lambda_2 \left( (H_1)_{i,j}^n + (H_2)_{i,j}^n \right) - \lambda_1 (E_3)_{i,j}^n,$$

(3) Method-III

$$\Phi^{[III]}(t_{n+1}) = \Phi^{[III]}(t_n) + \Delta x \Delta y \sum_{i,j} \left( \Upsilon_{i,j}^{n+\frac{1}{2}} (\Delta W)_{i,j}^n \right),$$

where

$$\Phi^{[III]}(t_{n+1}) = \Delta x \Delta y \sum_{i,j} \left[ \left| (E_3)_{i,j}^{n+1} \right|^2 + \left| (H_1)_{i,j}^{n+1} \right|^2 + \left| (H_2)_{i,j}^{n+1} \right|^2 \right]$$

and

$$\Upsilon_{i,j}^{n+\frac{1}{2}} = \lambda_2 \left( (H_1)_{i,j}^{n+\frac{1}{2}} + (H_2)_{i,j}^{n+\frac{1}{2}} \right) - \lambda_1 (E_3)_{i,j}^{n+\frac{1}{2}}.$$

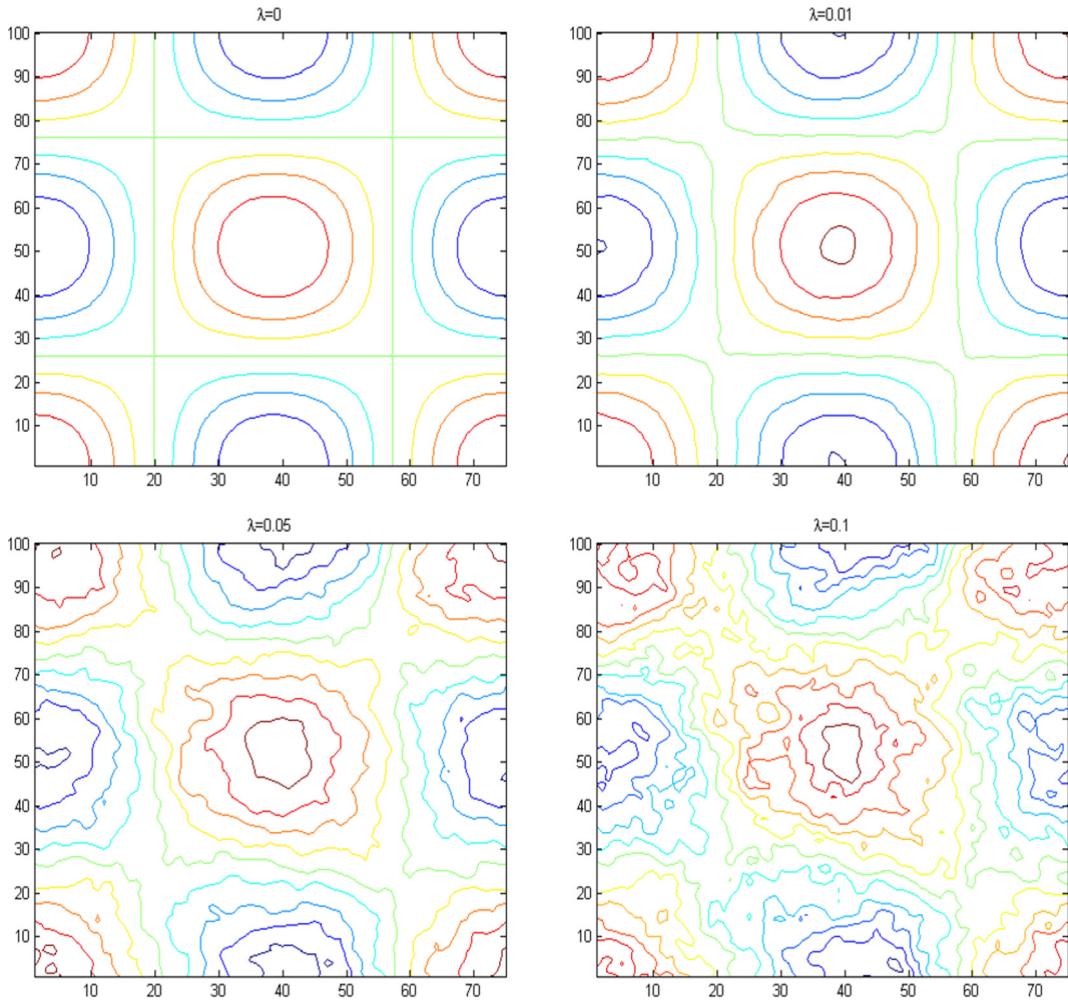


Fig. 4.2. Contours of the  $H_1$  for different sizes of noise  $\lambda = 0$ ,  $\lambda = 0.01$ ,  $\lambda = 0.05$  and  $\lambda = 0.1$ .

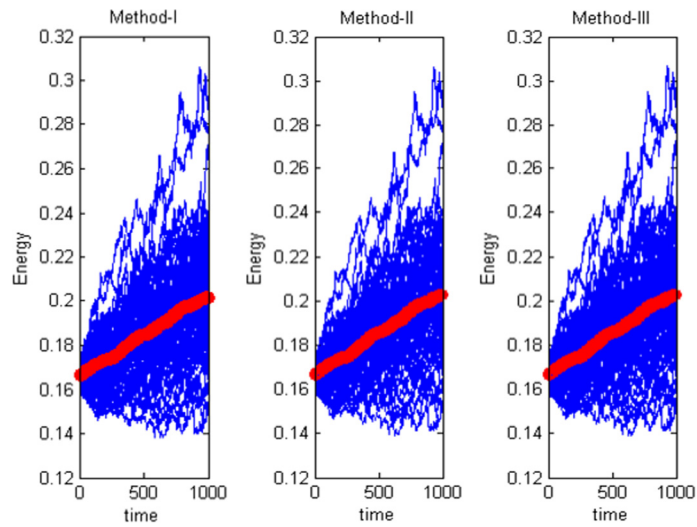
Fig. 4.3 presents the simulation of energies using the proposed methods in Section 3, where the blue lines denote discrete energies along 100 trajectories respectively, and the red lines represent the discrete averaged energies using Monte-Carlo method. From Theorem 2.1, we know that the linear increment slope of the averaged energy in the continuous case is  $K = 3(\lambda_1^2 + \lambda_2^2)Tr(Q)$ . As we take  $\lambda_1 = \lambda_2 = 0.1$  here, it follows that  $K \approx 0.0816$  (because of  $Tr(Q) \approx 1.36$ ), which leads to the averaged energy at time  $T = 1$  should be 0.2501. We may observe from Fig. 4.3 that the averaged energy (red line) is linear growth with respect to the time for all of three numerical methods. It extends the theoretical results for the estimation of the averaged energy in Section 3, since Theorem 3.3 tells that for time-dependent noise, the averaged energy evolves linearly and Theorem 3.11 states that for Method-III, the averaged energy evolves at most linearly. But the values of discrete averaged energy is 0.2 approximately at time  $T = 1$ , which is a bit smaller than the number 0.2501 of the continuous case. It may be caused by taking averaged value only over 100 paths, i.e.,  $\sum_{p=1}^{100} \frac{1}{100} \Phi(t_n, \omega_p) \approx \mathcal{E}(\Phi(t_n))$  with  $\Phi(t_n)$  being the discrete energy of one of three methods. As we will observe for the error of divergence; it should be zero theoretically, however, it is of  $10^{-2}$  numerically when we approximate it over 100 paths. Meanwhile, Fig. 4.4 presents the probability density functions of random variable  $\max_n \Phi(t_n)$  with  $\Phi(t_n)$  being the discrete energy of Method-I, Method-II and Method-III, respectively. We may observe that the probability density functions look similar for all of the three methods with slightly different probabilities.

Moreover, we consider the numerical simulation for the discrete conservation law of averaged divergence. Since the first two components of electric field  $\mathbf{E}$  are zero for 2-D system (4.1), which means that the averaged divergence-preserving property holds naturally. We consider that property of magnetic field  $\mathbf{H}$  in the following. The definitions of the corresponding discrete divergences are given as following:

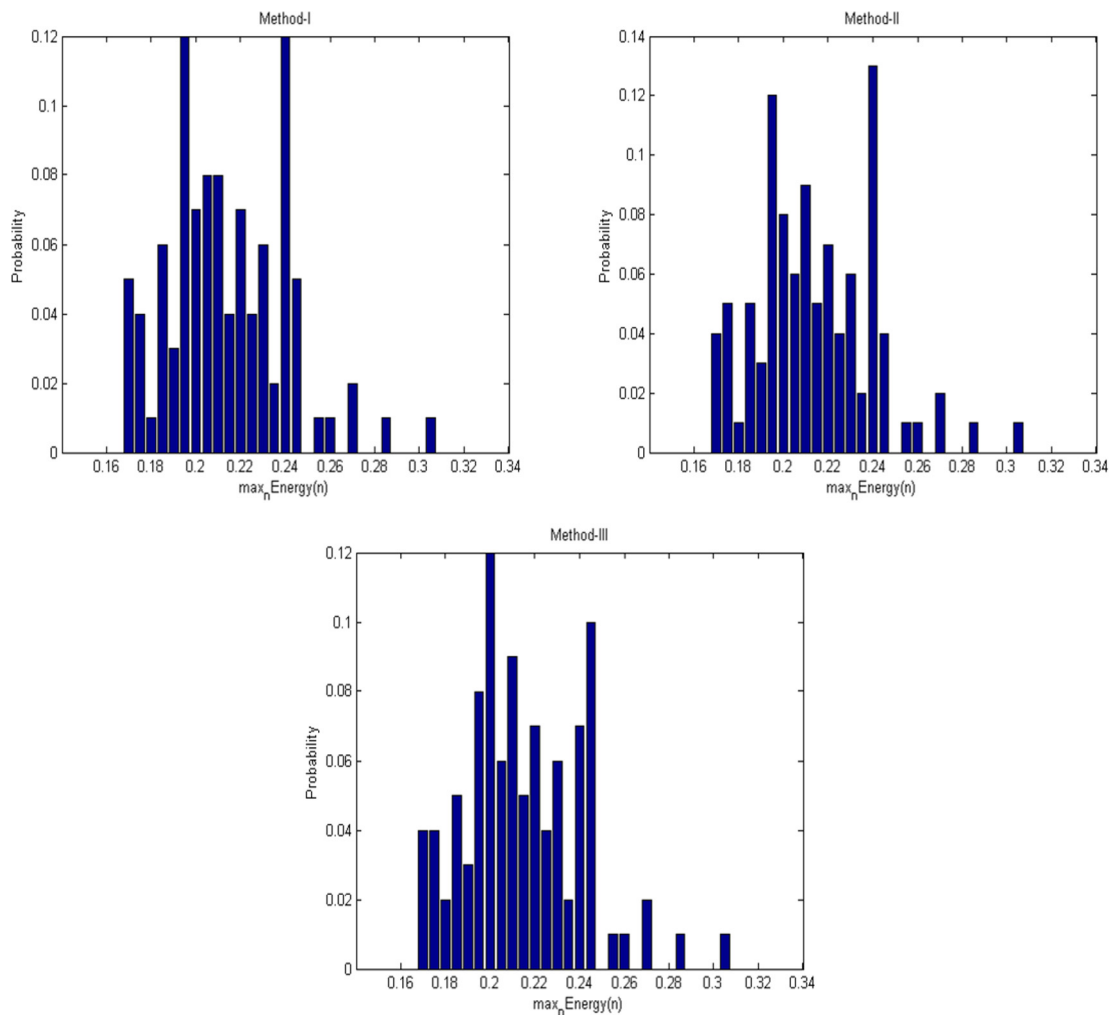
(1) Method-I

$$\bar{\nabla}_{i,j}^{[I]} \cdot \mathbf{H}^n = \delta_x(H_1)_{i-\frac{1}{2},j+\frac{1}{2}}^n + \delta_x(H_1)_{i-\frac{1}{2},j-\frac{1}{2}}^n + \delta_y(H_2)_{i+\frac{1}{2},j-\frac{1}{2}}^n + \delta_y(H_2)_{i-\frac{1}{2},j-\frac{1}{2}}^n,$$





**Fig. 4.3.** The averaged energy by Method-I (left), Method-II (middle) and Method-III (right) for  $\lambda_1 = \lambda_2 = 0.1$ . (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)



**Fig. 4.4.** The probability of density function of discrete energy by Method-I (top left), Method-II (top right) and Method-III (below) for  $\lambda_1 = \lambda_2 = 0.1$ .

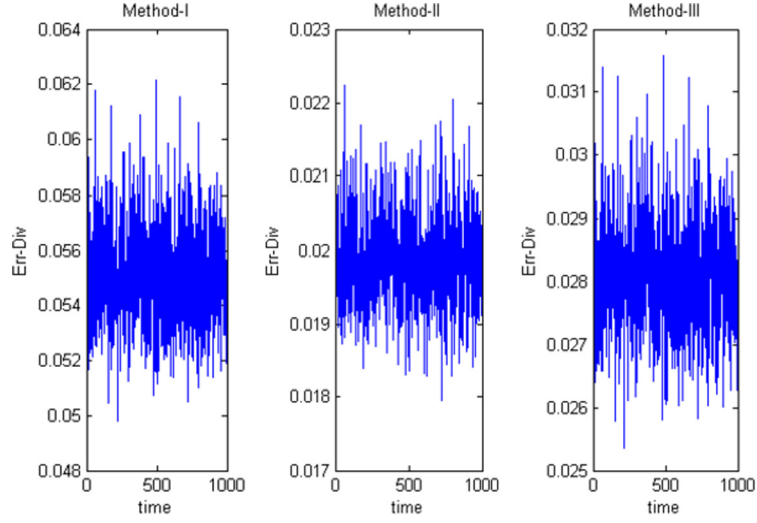


Fig. 4.5. The error of averaged divergence of Method-I (left), Method-II (middle) and Method-III (right) for  $\lambda_1 = \lambda_2 = 0.1$  and  $P = 100$ .

(2) Method-II

$$\bar{\nabla}^{[II]} \cdot \mathbf{H}_{i,j}^{n+\frac{1}{2}} = \bar{\delta}_x(H_1)_{i,j}^{n+\frac{1}{2}} + \bar{\delta}_y(H_2)_{i,j}^{n+\frac{1}{2}},$$

(3) Method-III

$$\bar{\nabla}^{[III]} \cdot \mathbf{H}_{i,j}^n = \bar{\delta}_x(H_1)_{i,j}^n + \bar{\delta}_y(H_2)_{i,j}^n.$$

We numerically perform the error of divergence by Monte Carlo method, which is defined by

$$\text{Err-Div}(n) = \Delta x \Delta y \sum_i \sum_j \left| \frac{1}{P} \sum_{s=1}^P \left( \bar{\nabla}_{i,j} \cdot \mathbf{H}^{n+1}(\omega_s) - \bar{\nabla}_{i,j} \cdot \mathbf{H}^n(\omega_s) \right) \right|.$$

The error results for three methods are displayed in Fig. 4.5. Observe that the scale of the error here is of  $10^{-2}$  for  $P = 100$ . This may be due to the value of  $P$  is only 100. This point of view is checked in the following. Thanks to the special structure of the error of divergence, which means they can be rewritten as the difference of Wiener increments, i.e.,

(1) Method-I

$$\begin{aligned} \bar{\nabla}_{i,j}^{[I]} \cdot \mathbf{H}^{n+1} - \bar{\nabla}_{i,j}^{[I]} \cdot \mathbf{H}^n &= \frac{\lambda_2}{\Delta x} \left[ (\Delta W)_{i,j}^n + (\Delta W)_{i,j-1}^n - (\Delta W)_{i-1,j}^n - (\Delta W)_{i-1,j-1}^n \right] \\ &\quad + \frac{\lambda_2}{\Delta y} \left[ (\Delta W)_{i,j}^n + (\Delta W)_{i-1,j}^n - (\Delta W)_{i,j-1}^n - (\Delta W)_{i-1,j-1}^n \right], \end{aligned}$$

(2) Method-II

$$\begin{aligned} \bar{\nabla}_{i,j}^{[II]} \cdot \mathbf{H}_{i,j}^{n+\frac{1}{2}} - \bar{\nabla}_{i,j}^{[II]} \cdot \mathbf{H}_{i,j}^{n-\frac{1}{2}} &= \frac{\lambda_2}{4\Delta x} \left[ (\Delta W)_{i+1,j}^n + (\Delta W)_{i+1,j}^{n-1} - (\Delta W)_{i-1,j}^n - (\Delta W)_{i-1,j}^{n-1} \right] \\ &\quad + \frac{\lambda_2}{4\Delta y} \left[ (\Delta W)_{i,j+1}^n + (\Delta W)_{i,j+1}^{n-1} - (\Delta W)_{i,j-1}^n - (\Delta W)_{i,j-1}^{n-1} \right], \end{aligned}$$

(3) Method-III

$$\bar{\nabla}_{i,j}^{[III]} \cdot \mathbf{H}_{i,j}^{n+1} - \bar{\nabla}_{i,j}^{[III]} \cdot \mathbf{H}_{i,j}^n = \frac{\lambda_2}{2\Delta x} \left[ (\Delta W)_{i+1,j}^n - (\Delta W)_{i-1,j}^n \right] + \frac{\lambda_2}{2\Delta y} \left[ (\Delta W)_{i,j+1}^n - (\Delta W)_{i,j-1}^n \right].$$

We can utilize the right-hand sides of the above equalities to perform the influence of the number of paths  $P$  without solving the equations themselves directly. Take  $\lambda_1 = \lambda_2 = 0.1$  in Method-III for an example. We take the number of trajectories  $P = 10, 10^2, 10^3, 10^4, 10^5, 10^6$  respectively to obtain the corresponding values of

$$\frac{1}{IJ} \sum_{i,j} \left| \frac{\lambda_2}{2\Delta x} \frac{1}{P} \sum_{s=1}^P \left( (\Delta W)_{i+1,j}^n - (\Delta W)_{i-1,j}^n \right) (\omega_s) + \frac{\lambda_2}{2\Delta y} \frac{1}{P} \sum_{s=1}^P \left( (\Delta W)_{i,j+1}^n - (\Delta W)_{i,j-1}^n \right) (\omega_s) \right|,$$

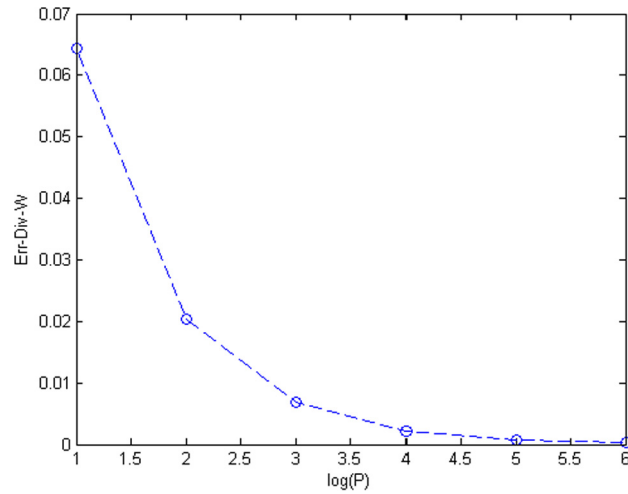


Fig. 4.6. The error of averaged divergence v.s. the number of trajectories of  $P = 10, 10^2, 10^3, 10^4, 10^5, 10^6$ .

Method-I			Method-III		
$\Delta t$	$(E \ err\ _{L^2}^2)^{1/2}$	Order	$\Delta t$	$(E \ err\ _{L^2}^2)^{1/2}$	Order
$2^{-6}$	0.0793	-	$2^{-6}$	0.0884	-
$2^{-7}$	0.0609	0.3805	$2^{-7}$	0.0648	0.4483
$2^{-8}$	0.0452	0.4296	$2^{-8}$	0.0424	0.6113

Fig. 4.7. Mean-square convergence order of Method-I and Method-III for  $\lambda_1 = \lambda_2 = 0.1$ .

which represent the error of averaged divergence. From the numerical result, we know that the global residuals of the discrete averaged divergence become smaller and smaller with the increasing of the number of trajectories  $P$  (see Fig. 4.6).

Finally, we consider the mean-square convergence orders of Method-I and Method-III from numerical point of view, because Method-II requires more restriction on the mesh sizes. Fix  $\lambda_1 = \lambda_2 = 0.1$  and  $\Delta x = \Delta y = \frac{1}{150}$ . Fig. 4.7 displays the convergence orders in mean-square sense, where

$$\|err\|_{L^2}^2 = \Delta x \Delta y \sum_{i,j} \left[ \left( (E_3)_{i,j}^N - (E_3^{ref})_{i,j}^N \right)^2 + \left( (H_1)_{i,j}^N - (H_1^{ref})_{i,j}^N \right)^2 + \left( (H_2)_{i,j}^N - (H_2^{ref})_{i,j}^N \right)^2 \right].$$

The reference solution is computed using the time step size  $\Delta t = 2^{-9}$  and the expectation is realized using the average of 100 independent paths. We may observe a mean-square order of convergence around 0.5 for Method-I and Method-III. It is interesting to investigate the convergence results theoretically.

### 5. Concluding remarks

In this paper, we firstly studied some properties of continuous system of stochastic Maxwell equations driven by an additive noise. By using a direct approach, we rewrite stochastic Maxwell equations into the form of stochastic Hamiltonian PDEs, and we show that they preserve stochastic multi-symplectic structure almost surely. Furthermore, it is shown that the averaged energy increases linearly with respect to the evolution of time, and divergence is preserved in the sense of expectation.

Secondly, we proposed and analyzed three stochastic multi-symplectic numerical methods to discretize stochastic Maxwell equations with additive noise. Our start point is that in deterministic model, for lossless media, energy is a conserved quantity and divergence is free with no free charges or currents. They are important criteria to evaluate a numerical method is good or not. As is shown in continuous stochastic case, the averaged energy evolves linearly with the growth of time which is caused by random source, and the divergence is preserved in the sense of expectation which means electric flux and magnetic flux are preserved in Gaussian random fields in the statistical sense. It is meaningful to investigate the preservation of these physical properties by the three numerical methods. We showed that the three numerical methods conserve the corresponding versions of dissipative properties of the averaged energy, and the discrete averaged energies evolve at most linearly with respect to time. For Method-I, we only obtain the linear evolution relationship for the case that the noise only depends on time variable; the result of Method-II approximates the continuous case best, for which we show that the discrete averaged energy evolves linearly with the rate approximates the one of continuous case for temporal-spatial noise; For Method-III, the situation is similar as that of Method-I, but furthermore, we show that for

the general noise case, the discrete averaged energy of Method-III evolves at most linearly. Moreover, the three methods preserve the conservation law of the discrete divergence in the sense of expectation.

At last, some numerical experiments are performed to support our theoretical results. To truncate the infinite-dimensional Wiener process, which might be represented as an infinite summation of a sequence, we display the values of the sequence with respect to indices. We observe that for small noise, the electric and magnetic waves are not strongly perturbed, but when the noise level is higher and apparently the waves are destroyed. In the performance of discrete averaged energy and divergence, we could observe from Section 4, all of the three methods reach the similar results. Furthermore, special attentions are needed to pay to the performance of Method-I, since the condition number of its iterates matrix is poorer than that of Method-II and Method-III, we utilize the splitting strategy proposed in [6] to deal with the problem, which could still preserve the discrete stochastic multi-symplectic conservation law. As for Method-II, it is a three-layer method, which needs another method to initialize, while the evolution of the discrete averaged energy is supported better in theoretical than Method-I and Method-III.

## References

- [1] M. Badieirostami, A. Adibi, H. Zhou, S. Chow, Model for efficient simulation of spatially incoherent light using the Wiener chaos expansion method, *Opt. Lett.* 32 (2007).
- [2] J. Hong, L. Ji, L. Zhang, A stochastic multi-symplectic scheme for stochastic Maxwell equations with additive noise, *J. Comput. Phys.* 268 (2014) 255–268.
- [3] J. Hong, L. Ji, L. Zhang, J. Cai, Stochastic multi-symplectic wavelet collocation method for 3D stochastic Maxwell equations, arXiv:1410.3552.
- [4] T. Horsin, I. Stratis, A. Yannacopoulos, On the approximate controllability of the stochastic Maxwell equations, *IMA J. Math. Control Inf.* 27 (2010) 103–118.
- [5] S. Jiang, L. Wang, J. Hong, Stochastic multi-symplectic integrator for stochastic Hamiltonian nonlinear Schrödinger equation, *Commun. Comput. Phys.* 14 (2013) 393–411.
- [6] L. Kong, J. Hong, J. Zhang, Splitting multi-symplectic integrators for Maxwell equations, *J. Comput. Phys.* 229 (2010) 4259–4278.
- [7] L. Kurt, T. Schaefer, Propagation of ultra-short solitons in stochastic Maxwell equations, *J. Math. Phys.* 55 (2014) 1–11.
- [8] J. Lee, B. Fornber, Some unconditionally stable time stepping methods for the 3D Maxwell's equations, *J. Comput. Appl. Math.* 166 (2004) 497–523.
- [9] K. Liaskos, I. Stratis, A. Yannacopoulos, Stochastic integrodifferential equations in Hilbert spaces with applications in electromagnetics, *J. Integral Equ. Appl.* 22 (2010) 559–590.
- [10] G. Liu, Stochastic wave propagation in Maxwell's equations, *J. Stat. Phys.* 158 (2015) 1126–1146.
- [11] G. Roach, I. Stratis, A. Yannacopoulos, *Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics*, Princeton University Press, 2012.
- [12] S. Rytov, Y. Kravtsov, V. Tatarskii, *Principles of Statistical Radiophysics: Elements of Random Fields 3*, Springer, Berlin, 1989.
- [13] Y. Sun, P. Tse, Symplectic and multi-symplectic numerical methods for Maxwell's equations, *J. Comput. Phys.* 230 (2011) 2076–2094.
- [14] S. Wen, Direct numerical simulation of near field thermal radiation based on Wiener chaos expansion of thermal fluctuation current, *J. Heat Transf.* 132 (2010).
- [15] K. Yee, Numerical solution of initial boundary value problem involving Maxwell equations in isotropic media, *IEEE Trans. Antennas Propag.* 14 (1966) 302–307.